Random effects models

AD Model Builder foundation

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About random effect models

- In purely fixed effects models we have
  - Random variables we observe
  - Model parameters we want to estimate
- In random effects models we have
  - Random variables we observe
  - Random variables we do NOT observe
  - Model parameters we want to estimate
- This model class is very useful and goes by many names: random effects models, mixed models, latent variable models, state-space models, frailty models, hierarchical models, ...
- Many tools can handle linear Gaussian models.
- No other tool handles non-linear non-Gaussian random effect models like ADMB

Example: Paired observations

- Two methods A and B to measure blood cell count (to check for the use of doping).
- Paired study.

Person ID	Method A	Method B
1	5.5	5.4
2	4.4	4.9
3	4.6	4.5
4	5.4	4.9
5	7.6	7.2
6	5.9	5.5
7	6.1	6.1
8	7.8	7.5
9	6.7	6.3
10	4.7	4.2

- It must be expected that two measurements from the same person are correlated, so a paired t-test is the correct analysis
- The t-test gives a p-value of 5.1%, which is a borderline result...
- But more data is available

- In addition to the planned study 10 persons were measured with only one method
- Want to use all data, which is possible with random effects
- Assume these 20 are ramdomly selected from a population where the blod cell count is normally distributed
- Consider the following model:

  $C_i = \alpha(M_i) + B(P_i) + \varepsilon_i, \quad i = 1 \ldots 30$

  $\alpha(M_i)$ the 2 fixed method effects

  $B(P_i) \sim \mathcal{N}(0, \sigma_P^2)$ the 20 random effects

  $\varepsilon_i \sim \mathcal{N}(0, \sigma_R^2)$ measurement noise

  All $B(P_i)$ and ε_i are assumed independent
- This model uses all data and gives a 95% c. i. for the method bias $\alpha(A) - \alpha(B)$ which is: $(0.04; 0.41)$.
- Notice that now there is a (slightly) significant method bias.

Person ID	**Method A**	**Method B**
1	5.5	5.4
2	4.4	4.9
3	4.6	4.5
4	5.4	4.9
5	7.6	7.2
6	5.9	5.5
7	6.1	6.1
8	7.8	7.5
9	6.7	6.3
10	4.7	4.2
11		5.1
12		4.4
13		4.5
14		5.3
15		7.5
16	5.7	
17	6.0	
18	7.5	
19	6.5	
20	4.2	

```
#No rows
 30
#No cols
 3
#The obs matrix
#P M C
1 1 5.5
2 1 4.4
3 1 4.6
4 1 5.4
5 1 7.6
6 1 5.9
7 1 6.1
8 1 7.8
9 1 6.7
10 1 4.7
16 1 5.7
17 1 6
18 1 7.5
19 1 6.5
20 1 4.2
1 2 5.4
2 2 4.9
3 2 4.5
4 2 4.9
5 2 7.2
6 2 5.5
7 2 6.1
8 2 7.5
9 2 6.3
10 2 4.2
11 2 5.1
12 2 4.4
13 2 4.5
14 2 5.3
15 2 7.5
```

```
DATA_SECTION
  init_int nrow;
  init_int ncol;
  init_matrix obs(1,nrow,1,ncol);

  vector C(1,nrow);
  ivector P(1,nrow);
  ivector M(1,nrow);

  !! C=column(obs,3);
  !! P=(ivector)column(obs,1);
  !! M=(ivector)column(obs,2);
PARAMETER_SECTION
  init_number logSigmaP;
  init_number logSigmaR;
  init_vector alpha(1,2);

  random_effects_vector B(1,20);

  sdreport_number sigmaP;
  sdreport_number sigmaR;
  sdreport_number diffAB;
  vector pred(1,nrow);
  objective_function_value nll;
PROCEDURE_SECTION
  sigmaR=exp(logSigmaR);
  sigmaP=exp(logSigmaP);
  dvariable ss;

  nll=0.0;
  ss=square(sigmaR);
  for(int i=1; i<=nrow; ++i){
    pred(i)=alpha(M(i))+B(P(i));
    nll+=0.5*(log(2*M_PI*ss)+square(C(i)-pred(i))/ss);
  }
  ss=square(sigmaP);
  for(int i=1; i<=20; ++i){
    nll+=0.5*(log(2*M_PI*ss)+square(B(i))/ss);
  }
  diffAB=alpha(1)-alpha(2);
```

Random effects in AD Model Builder

- In random effects models we have
  - Random variables we observe: x
  - Random variables we do not observe: z
  - Model parameters we want to estimate: θ
- If we had observed x and z we would have a joint likelihood $L(x, z, \theta)$
- but z is unobserved so we have to estimate θ in the marginal likelihood:

$$L(x, \theta) = \int L(x, z, \theta) dz$$

- This requires a high dimensional integral — which is difficult
- This is (part of) the reason MCMC methods are so widely used
- MCMC can be slow, difficult to judge convergence, and in tools like winBugs a prior must be assigned to everything — even when you have no prior information.
- AD Model Builder has a better solution

Laplace approximation

- Want to compute the marginal likelihood for a given θ value:

$$L(x,\theta) = \int L(x,z,\theta)dz$$

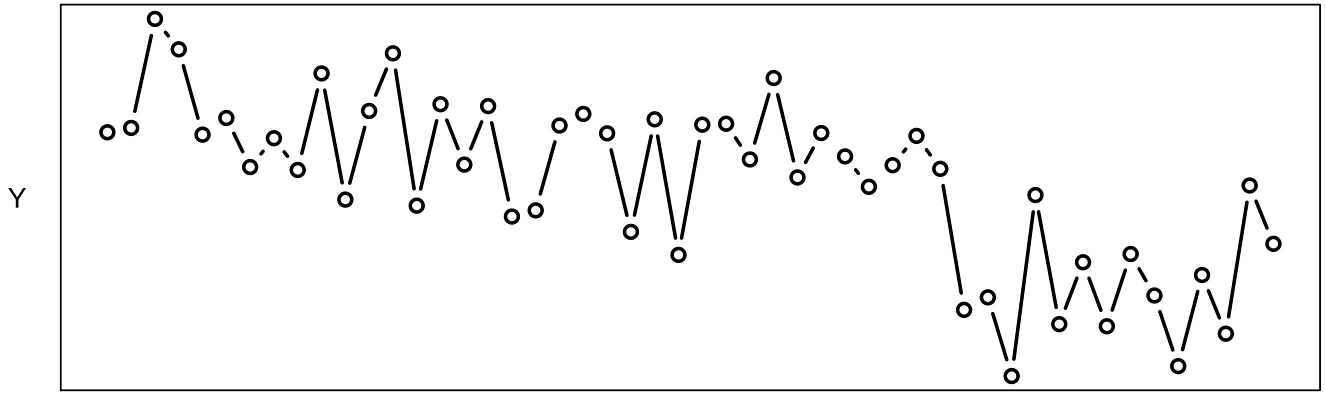
- First the joint likelihood $L(x,z,\theta)$ is optimized w.r.t. z.
- This optimization yields an estimate $\hat{z}$, and an estimated hessian $\mathcal{H}(\hat{z})$.
- Next a Gaussian approximation is assumed and the result (apart from a constant) is:

$$L(x,\theta) \approx |\det(\mathcal{H}(\hat{z}))|^{-0.5} L(x,\hat{z},\theta)$$

- Notice that when defined in this way $\hat{z}$ and $\mathcal{H}(\hat{z})$ and also depend on θ, which makes AD of this pretty difficult, but all solved for us in AD Model Builder.
- Actually this is all very simple to use. All we have to do is:
  - Code up the joint negative log likelihood
  - declare as `random_effects_vector z(1,n);`

Example: Estimating latent random walk

- Observation vector Y generated from:
  - $\lambda_i = \lambda_{i-1} + \eta_i$
  - $Y_i = \lambda_i + \varepsilon_i$
  - where $i = 1 \ldots 50$, $\eta_i \sim \mathcal{N}(0, \sigma_\lambda^2)$, and $\varepsilon_i \sim \mathcal{N}(0, \sigma_Y^2)$ all independent.

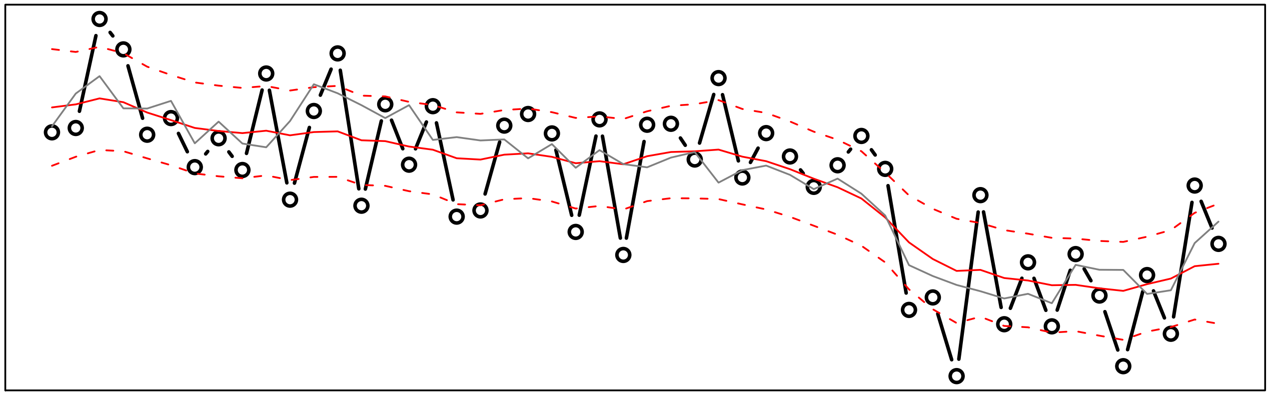


  - Notice λ vector unobserved, and here we wish to estimate λ
- Knowing what we know now — how should we model this?
- Consider λ as unobserved random variable
  - Estimate model parameters (σ_λ and σ_ε) in marginal distribution $\int p(\lambda, Y) d\lambda$
  - Predict λ via distribution of $\lambda | Y$

```
DATA_SECTION
  init_int N
  init_vector y(1,N)

PARAMETER_SECTION
  init_number logSdLam
  init_number logSdy
  random_effects_vector lam(1,N);
  objective_function_value jnll;

PROCEDURE_SECTION
  jnll=0.0;
  dvariable var;
  var=exp(2.0*logSdLam);
  for(int i=2; i<=N; ++i){
    jnll+=0.5*(log(2.0*M_PI*var)
               +square(lam(i)-lam(i-1))/var);
  }
  var=exp(2.0*logSdy);
  for(int i=1; i<=N; ++i){
    jnll+=0.5*(log(2.0*M_PI*var)
               +square(lam(i)-y(i))/var);
  }

TOP_OF_MAIN_SECTION
  gradient_structure::set_MAX_NVAR_OFFSET(3000);
```

index	name	value	std dev
1	logSdLam	-2.3576e-01	3.3713e-01
2	logSdy	8.1161e-01	1.1955e-01
3	lam	9.4885e-01	1.2231e+00
4	lam	1.0772e+00	1.0988e+00
5	lam	1.3274e+00	1.0810e+00
6	lam	1.1676e+00	1.0275e+00
7	lam	7.3510e-01	9.6103e-01
8	lam	4.1696e-01	9.4607e-01
9	lam	8.8186e-02	9.5341e-01
10	lam	-3.8547e-02	9.5028e-01
.			
.			
.			
43	lam	-6.2033e+00	1.0043e+00
44	lam	-6.3070e+00	9.8098e-01
45	lam	-6.5045e+00	9.9328e-01
46	lam	-6.4900e+00	9.7044e-01
47	lam	-6.6344e+00	9.9289e-01
48	lam	-6.7417e+00	1.0266e+00
49	lam	-6.4604e+00	9.9465e-01
50	lam	-6.2260e+00	1.0168e+00
51	lam	-5.7070e+00	1.1180e+00
52	lam	-5.6044e+00	1.2585e+00

Y



More efficient coding

```
DATA_SECTION
  init_int N
  init_vector y(1,N)

PARAMETER_SECTION
  init_number logSdLam
  init_number logSdy
  random_effects_vector lam(1,N);
  objective_function_value jnll;

PROCEDURE_SECTION
  jnll=0.0;
  dvariable var;

  for(int i=2; i<=N; ++i){
    step(lam(i-1),lam(i),logSdLam);
  }

  for(int i=1; i<=N; ++i){
    obs(lam(i),logSdy,i);
  }
```

- The idea is to reduce the likelihood calculation to a sum of function calls, where each call only uses a few random effects.
- Each function call must include the parameters needed, and the random effects needed, and not much more (no need to pass data)
- Function headers must be one line — even when they get too long.

```
SEPARABLE_FUNCTION void step(const dvariable& lam1, const dvariable& lam2, const dvariable& logSdLam)
  dvariable var=exp(2.0*logSdLam);
  jnll+=0.5*(log(2.0*M_PI*var)+square(lam2-lam1)/var);

SEPARABLE_FUNCTION void obs(const dvariable& lam, const dvariable& logSdy, int i)
  dvariable var=exp(2.0*logSdy);
  jnll+=0.5*(log(2.0*M_PI*var)+square(lam-y(i))/var);

TOP_OF_MAIN_SECTION
  gradient_structure::set_MAX_NVAR_OFFSET(3000);
```

Example: Discrete valued time series

- One of the examples from the AD Model Builder site (from Kuk & Cheng (1999)).
- The model:

$$y_i \sim \text{Pois}(\lambda_i)\ , \ \text{where}$$
$$\log(\lambda_i) = X_i b + u_i\ , \ \text{and}$$
$$u_i = a u_{i-1} + \varepsilon_i$$

Here, X_i is a covariate vector, b is a vector of regression parameters and u_i is an AR(1). The dimension of b is 6 and i=1,. . . ,168.

	β_1	β_2	β_3	β_4	β_5	β_6	a	σ
ADMB-RE	0.242	-3.81	0.162	-0.482	0.413	-0.0109	0.627	0.538
Std. dev.	0.270	2.76	0.15	0.16	0.13	0.13	0.19	0.15
Kuk & Cheng	0.244	-3.82	0.162	-0.478	0.413	-0.0109	0.665	0.519

Discrete valued time series code

```
DATA_SECTION
  init_int n
  init_vector y(1,n)
  init_int p
  init_matrix X(1,n,1,p)
PARAMETER_SECTION
  init_vector b(1,p,1)
  init_bounded_number a(-1,1,2)
  init_number log_sigma(2)
  random_effects_vector u(1,n,2)
  objective_function_value g
PROCEDURE_SECTION
  g=0.0; int i;

  sf1(log_sigma,a,u(1));

  for (i=2;i<=n;i++){
    sf2(log_sigma,a,u(i),u(i-1),i);
  }

  for (i=1;i<=n;i++){
    sf3(u(i),b,i);
  }
SEPARABLE_FUNCTION void sf1(const dvariable& ls,const dvariable& aa,const dvariable& u_1)
  g += ls - 0.5*log(1-square(aa)) +0.5*square(u_1/exp(ls))*(1-square(aa));
SEPARABLE_FUNCTION void sf2(const dvariable& ls, const dvariable& aa,const dvariable& u_i,const dvaria
  g += ls +.5*square((u_i-aa*u_i1)/exp(ls));
SEPARABLE_FUNCTION void sf3(const dvariable& u_i ,const dvar_vector& bb, int i)
  dvariable eta = X(i)*bb + u_i;
  dvariable lambda = exp(eta);
  g -= y(i)*eta - lambda;
TOP_OF_MAIN_SECTION
  gradient_structure::set_MAX_NVAR_OFFSET(1000);
```

Non-Gaussian random effects

- If the random effects are non-Gaussian, then the Laplace approximation may be inaccurate.
- Can use transformation $g = F^{-1}(\Phi(u))$, where $u \sim \mathcal{N}(0, 1)$.
- E.g. part of a larger example:

```
...
PARAMETER_SECTION
  ...
  random_effects_vector u(1,nh,2)
PROCEDURE_SECTION
  ...
  for (i=1;i<=nh;i++){
    fun(i,j,u(i),log_theta1,beta);
  }
SEPARABLE_FUNCTION void fun( int i,int & j ,const prevariable& ui, const prevariable& log_theta1,
 f += 0.9189385 + 0.5*square(ui);                    // N(0,1) likelihood contribution from u's
 ...
 dvariable z=cumd_norm(ui);                          // z has uniform (0,1) distribution
 z = 0.99999999*z + 0.000000005;                     // To gain numerical stability
 dvariable gi = theta1*inv_cumd_gamma(z,1.0/theta1);
...
```

- In situations where we fear the Laplace approximation may be inaccurate, we can improve it by importance sampling. Simply by:

```
./model -is 100
```

REML via random effects?

- The following non-linear model is assumed to describe the relation between density D within pot and yield Y per plant:

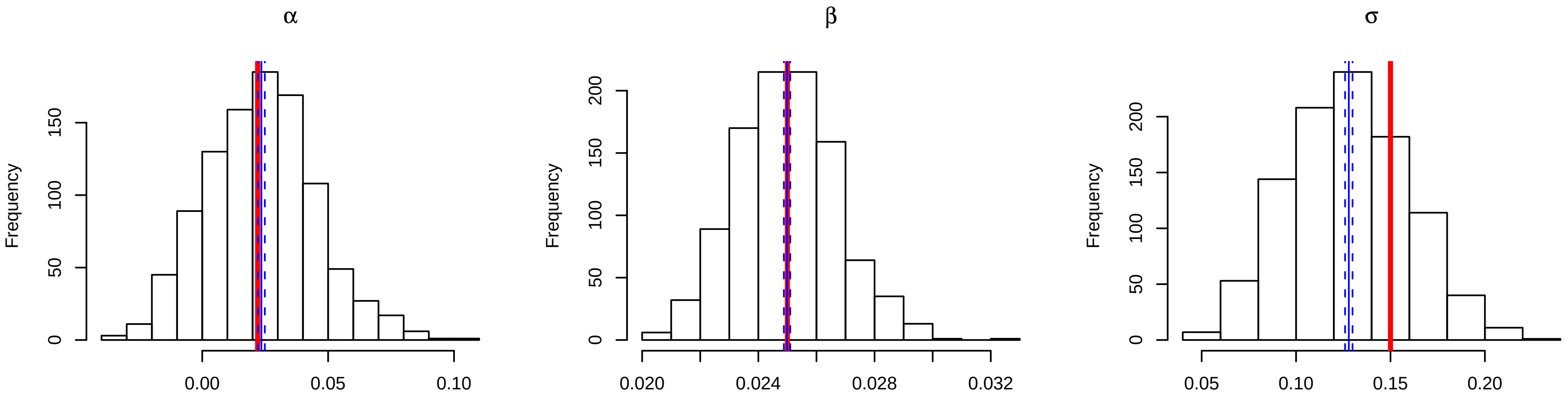
$$\log(Y_i) = -\log(\alpha + \beta D_i) + \varepsilon_i, \quad \text{where } \varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

```
DATA_SECTION
  init_int N
  init_vector density(1,N)
  init_vector yield(1,N)
  vector logYield(1,N)
  !! logYield=log(yield);
PARAMETER_SECTION
  init_number logA
  init_number logB
  init_number logSigma
  objective_function_value nll
  sdreport_number a
  sdreport_number b
  sdreport_number sigma
  vector pred(1,N)
  number ss
PROCEDURE_SECTION
  b=exp(logB);
  a=exp(logA)-b*min(density);
  sigma=exp(logSigma);
  ss=square(sigma);
  pred=-log(a+b*density);
  nll=0.5*(N*log(2*M_PI*ss)+
   sum(square(logYield-pred))/ss);
```

```
#N
10
#density
5 7 10 15 25 34 51 77 115 173
#yield
6.97 5.569 2.814 2.401 1.89 1.124 0.623 0.592 0.382 0.204
```

```
index   name      value       std dev
    1   logA      -1.9044e+00 1.0966e-01
    2   logB      -3.6793e+00 6.7797e-02
    3   logSigma  -1.9246e+00 2.2361e-01
    4   a          2.2708e-02 2.1107e-02
    5   b          2.5241e-02 1.7113e-03
    6   sigma      1.4594e-01 3.2633e-02
```

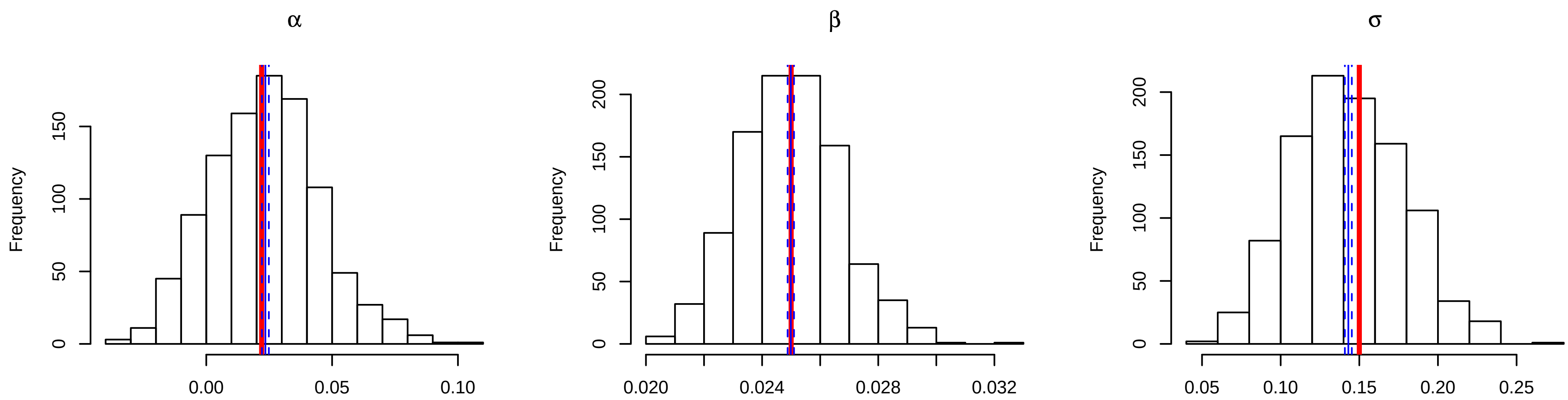
Simulation study



Parameter	True value	low	high
α	0.022	0.022	0.025
β	0.025	0.0249	0.0251
σ	0.150	0.126	0.13

This is an expected result, but can we fix it?

Results with random effects (flat prior) on α and β



Parameter	True value	low	high
α	0.022	0.022	0.025
β	0.025	0.0249	0.0251
σ	0.150	0.141	0.145

Almost!

What else?

- Large collection of examples at `http://www.otter-rsch.com/admbre/examples.html`
- It is possible to add priors on parameters (see exercise)
- The quality of the approximation can be checked and improved by importance sampling — without additional coding!
- Lots of flags for optimizing performance e.g. memory options — see manual.
- ...

Exercise: λ_0 in the latent random walk example

- As you may have noticed the model in the latent random walk example was not fully specified, as λ_0 was part of the model, but never defined.
- Here that is equivalent with assuming it has a uniform prior on $(-\infty, \infty)$.

? Estimate λ_0 via pure maximum likelihood estimation

? Estimate λ_0 with a Bayesian prior of $\lambda_0 \sim \mathcal{N}(0, 1)$

- Data for this exercise is:

```
#No obs
 50
#Y
 -0.09399342 0.08762907 4.657932 3.38314 -0.1941568 0.5034158 -1.553094 -0.3431696
-1.673901 2.372934 -2.917300 0.8004703 3.21504 -3.170574 1.081191 -1.449991
1.001843 -3.627856 -3.369206 0.1883197 0.6740543 -0.1392156 -4.269124 0.4490485
-5.234534 0.2239184 0.2639806 -1.233715 2.179709 -1.988403 -0.1270127 -1.106568
-2.379884 -1.475134 -0.2455092 -1.625744 -7.538624 -7.015322 -10.31427 -2.727188
-8.139333 -5.544363 -8.227553 -5.198673 -6.936379 -9.898509 -6.07848 -8.538303
-2.325157 -4.770373
```

Solution

```
DATA_SECTION
  init_int N
  init_vector y(1,N)

PARAMETER_SECTION
  init_number logSdLam
  init_number logSdy
  init_number lam0;//--- add model parameter---
  random_effects_vector lam(1,N);
  objective_function_value jnll;

PROCEDURE_SECTION
  jnll=0.0;
  dvariable var;
  var=exp(2.0*logSdLam);
                       //--- Include it in jnll ---
  jnll+=0.5*(log(2.0*M_PI*var)
             +square(lam(1)-lam0)/var);

  for(int i=2; i<=N; ++i){
    jnll+=0.5*(log(2.0*M_PI*var)
               +square(lam(i)-lam(i-1))/var);
  }

  var=exp(2.0*logSdy);
  for(int i=1; i<=N; ++i){
    jnll+=0.5*(log(2.0*M_PI*var)
               +square(lam(i)-y(i))/var);
  }

TOP_OF_MAIN_SECTION
  gradient_structure::set_MAX_NVAR_OFFSET(3000);
```

```
DATA_SECTION
  init_int N
  init_vector y(1,N)

PARAMETER_SECTION
  init_number logSdLam
  init_number logSdy
  init_number lam0;
  random_effects_vector lam(1,N);
  objective_function_value jnll;

PROCEDURE_SECTION
  jnll=0.0;
  dvariable var;

             //--- Add prior N(0,1) to lam0 ---
  jnll+=0.5*(log(2.0*M_PI)+square(lam0));

  var=exp(2.0*logSdLam);
  jnll+=0.5*(log(2.0*M_PI*var)
             +square(lam(1)-lam0)/var);

  for(int i=2; i<=N; ++i){
    jnll+=0.5*(log(2.0*M_PI*var)
               +square(lam(i)-lam(i-1))/var);
  }

  var=exp(2.0*logSdy);
  for(int i=1; i<=N; ++i){
    jnll+=0.5*(log(2.0*M_PI*var)
               +square(lam(i)-y(i))/var);
  }

TOP_OF_MAIN_SECTION
  gradient_structure::set_MAX_NVAR_OFFSET(3000);
```

Exercise: Random effect logistic regression

- Read through the example at:
  http://mathstat.helsinki.fi/openbugs/Examples/Seeds.html

? Implement the same model in ADMB, but without priors on the hyper parameters.

? Compare results.

- The data for this exercise is:

```
#N
 21
#r
 10 23 23 26 17 5 53 55 32 46 10 8 10 8 23 0 3 22 15 32 3
#n
 39 62 81 51 39 6 74 72 51 79 13 16 30 28 45 4 12 41 30 51 7
#x1
 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1
#x2
 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1
```

Solution

```
DATA_SECTION
  init_int N;
  init_ivector r(1,N);
  init_ivector n(1,N);
  init_ivector x1(1,N);
  init_ivector x2(1,N);

PARAMETER_SECTION
  init_number alpha0
  init_number alpha1
  init_number alpha2
  init_number alpha12
  init_number logSigma

  random_effects_vector B(1,N)

  sdreport_number sigma
  vector logitp(1,N)
  vector p(1,N)

  objective_function_value jnll

PROCEDURE_SECTION
  sigma=exp(logSigma);
  logitp=alpha0+alpha1*x1+alpha2*x2+alpha12*elem_prod(x1,x2)+B;
  p=elem_div(exp(logitp),(1.0+exp(logitp)));

  jnll=0.0;
  for(int i=1; i<=N; ++i){
    jnll+=log_comb(n(i),r(i))-log(p(i))*r(i)-log(1.0-p(i))*(n(i)-r(i));
    jnll+=0.5*(log(2.0*M_PI*square(sigma))+square(B(i)/sigma));
  }
```

```
index   name      value       std dev
    1   alpha0    -5.4849e-01 1.6611e-01
    2   alpha1     9.7427e-02 2.7739e-01
    3   alpha2     1.3368e+00 2.3623e-01
    4   alpha12   -8.1003e-01 3.8422e-01
    5   logSigma  -1.4497e+00 4.6691e-01
    6   B         -1.5854e-01 2.2444e-01
    7   B          9.0476e-03 1.8999e-01
    8   B         -1.8273e-01 2.0536e-01
    9   B          2.4140e-01 2.3428e-01
   10   B          9.9434e-02 2.0804e-01
   11   B          4.5013e-02 2.3020e-01
   12   B          6.2799e-02 1.8985e-01
   13   B          1.6606e-01 2.0487e-01
   14   B         -1.0436e-01 2.0653e-01
   15   B         -2.3195e-01 2.2041e-01
   16   B          5.0781e-02 2.2308e-01
   17   B          8.0648e-02 2.2664e-01
   18   B         -6.6290e-02 2.1167e-01
   19   B         -1.1708e-01 2.2198e-01
   20   B          1.8894e-01 2.3589e-01
   21   B         -8.1461e-02 2.3978e-01
   22   B         -1.5248e-01 2.4853e-01
   23   B          2.5509e-02 2.0398e-01
   24   B         -2.2122e-02 2.0649e-01
   25   B          1.7960e-01 2.2983e-01
   26   B         -3.1760e-02 2.2604e-01
   27   sigma      2.3463e-01 1.0955e-01
```

Winbugs code

```
model{
    for( i in 1 : N ) {
       r[i] ~ dbin(p[i],n[i])
       b[i] ~ dnorm(0.0,tau)
       logit(p[i]) <- alpha0 + alpha1 * x1[i] + alpha2 * x2[i] +
          alpha12 * x1[i] * x2[i] + b[i]
    }
    alpha0 ~ dnorm(0.0,1.0E-6)
    alpha1 ~ dnorm(0.0,1.0E-6)
    alpha2 ~ dnorm(0.0,1.0E-6)
    alpha12 ~ dnorm(0.0,1.0E-6)
    tau ~ dgamma(0.001,0.001)
    sigma <- 1 / sqrt(tau)
}

list(r = c(10, 23, 23, 26, 17, 5, 53, 55, 32, 46, 10, 8, 10, 8, 23, 0, 3, 22, 15, 32, 3),
     n = c(39, 62, 81, 51, 39, 6, 74, 72, 51, 79, 13, 16, 30, 28, 45, 4, 12, 41, 30, 51, 7),
     x1 = c(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1,  1, 1, 1),
     x2 = c(0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1,  1, 1, 1),
     N = 21
)

list(alpha0 = 0, alpha1 = 0, alpha2 = 0, alpha12 = 0, tau = 10)
```